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# On the Monge–Ampère equivalent of the sine–Gordon equation

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**Abstract.** Surfaces of constant negative curvature in Euclidean space can be described by either the sine–Gordon equation for the angle between asymptotic directions, or a Monge–Ampère equation for the graph of the surface. We present the explicit form of the correspondence between these two integrable nonlinear partial differential equations using their well known properties in differential geometry. We find that the cotangent of the angle between asymptotic directions is directly related to the mean curvature of the surface. This is a Bäcklund-type transformation between the sine–Gordon and Monge–Ampère equations.

## 1. Introduction

Monge–Ampère equations have so far been excluded from extensive searches for integrable nonlinear evolution equations [1]. However, their origin lies in geometry and quite frequently we find them in the same category of problems where other well known integrable systems also find their natural setting. Some of these Monge–Ampère equations were listed in [2] but the identification of the Monge–Ampère equivalent of completely integrable nonlinear evolution equations is still an outstanding problem in many interesting cases. Perhaps the most remarkable of all such correspondences was established by Jörgens [3] who showed that the elliptic Monge–Ampère equation with one on right-hand side is equivalent to the equation governing minimal surfaces. For the hyperbolic Monge–Ampère equation the corresponding equation is the Born–Infeld equation, or the Euler equations for Chaplygin gas [4].

We shall now present the Monge–Ampère equivalent of the sine–Gordon equation. It is well known that

$$u_{xx}u_{yy} - u_{xy}^2 = -K^2 \quad K \equiv 1 + u_x^2 + u_y^2 \quad (1)$$

where  $z = u(x, y)$  is the graph of the surface, and the sine–Gordon equation

$$\phi_{\xi\eta} = \sin \phi \quad (2)$$

where  $\phi(\xi, \eta)$  is the angle between asymptotic directions must be related as they both govern surfaces of constant negative curvature. This observation occupies a central role in Anderson and Ibragimov's [5] exposition of the Bianchi–Lie transformation itself, as in their discussion they revert to either one of equations (1) or (2) freely even without presenting an

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explicit transformation between them. We shall now show that in the geometrical context the relation between  $\phi$  and  $u$  can be obtained directly. The result, given by (9) below, is an expression for the cotangent of the angle between asymptotic directions in terms of the mean curvature of the surface.

## 2. Geometrical preliminaries

We recall the first and second fundamental forms of a surface in  $E^3$  with curvature  $-1$  for which the Gauss–Codazzi equations give rise to the Monge–Ampère (1) and sine–Gordon (2) equations respectively. For the Monge–Ampère equation they are given by

$$\begin{aligned} ds_1^2 &= (1 + u_x^2) dx^2 + 2u_x u_y dx dy + (1 + u_y^2) dy^2 \\ ds_2^2 &= \frac{1}{\sqrt{K}} (u_{xx} dx^2 + 2u_{xy} dx dy + u_{yy} dy^2) \end{aligned} \quad (3)$$

while for the sine–Gordon case we have

$$\begin{aligned} ds_1^2 &= d\xi^2 + 2 \cos \phi d\xi d\eta + d\eta^2 \\ ds_2^2 &= 2\phi_{\xi\eta} d\xi d\eta. \end{aligned} \quad (4)$$

In order to establish the equivalence between equations (1) and (2) we shall require that their first and second fundamental forms must agree.

To this end we note that the  $\xi, \eta$  entering into the sine–Gordon equation are the asymptotic coordinates on the surface and we must first find their counterparts in the case of the Monge–Ampère equation. The characteristics of (1) satisfy

$$u_{xx}x'^2 + 2u_{xy}x'y' + u_{yy}y'^2 = 0 \quad (5)$$

which can be written in the form

$$[u_{xx}x' + (u_{xy} + K)y'][(u_{xy} + K)x' + u_{yy}y'] = 0 \quad (6)$$

and therefore we can introduce  $\xi, \eta$  such that

$$\begin{aligned} d\xi &= A [u_{xx}dx + (u_{xy} + K)dy] \\ d\eta &= B [(u_{xy} + K)dx + u_{yy}dy] \end{aligned} \quad (7)$$

where  $A, B$  may depend on  $u$  and its derivatives. The requirement that the first fundamental forms in (3) and (4) must agree yields

$$\begin{aligned} A^2 &= \frac{(1 + u_x^2)u_{yy}^2 - 2u_x u_y u_{yy}(u_{xy} + K) + (1 + u_y^2)(u_{xy} + K)^2}{4K^2(u_{xy} + K)^2} \\ B^2 &= \frac{(1 + u_y^2)u_{xx}^2 - 2u_x u_y u_{xx}(u_{xy} + K) + (1 + u_x^2)(u_{xy} + K)^2}{4K^2(u_{xy} + K)^2} \\ AB \cos \phi &= -\frac{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}}{4K^2(u_{xy} + K)} \end{aligned} \quad (8)$$

and these relations are sufficient to establish the identity of the second fundamental forms as well, cf equation (9) below.

The exterior derivative of (7) must vanish. Hence it follows that the coefficients of the 1-forms on the right-hand side of these equations are conserved quantities for the Monge–Ampère equation (1). This can be verified by a straightforward but lengthy calculation which also serves to establish the local existence of  $\xi$  and  $\eta$ .

### 3. The correspondence

Equations (8), which ensure the agreement of the first and second fundamental forms for the Monge–Ampère and the sine–Gordon equations, result in a Bäcklund-type transformation between  $\phi$  and  $u$ . This is most conveniently expressed in the form

$$\cot \phi = -\frac{1}{2K^{3/2}} \left[ (1 + u_y^2) u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2) u_{yy} \right] \quad (9)$$

which is simply the mean curvature of the surface obtained by the contraction of the first and second fundamental forms in (3). Remembering that  $\sqrt{K}$  is the coefficient of the volume element which also serves as the Lagrangian for minimal surfaces we can also write

$$\cot \phi = \frac{1}{2} \frac{\delta}{\delta u} \sqrt{K} \quad (10)$$

where  $\delta$  denotes the variational derivative. But the easiest way of remembering the result (9) is through its geometrical meaning, namely, the cotangent of the angle between asymptotic directions is minus one half the mean curvature of the surface.

### 4. Remarks

In the discussion of the existence of  $\xi, \eta$  through (7) we found integrals of the Monge–Ampère equation (1) that vanish along the characteristics. In general these are called *generalized flow functions*, or *characteristic integrals*. Vessiot [6] has presented a complete description of Monge–Ampère equations which admit infinitely many generalized flow functions. This list coincides with Zvyagin’s list [7] of second-order equations reducible to the wave equation by a Bäcklund transformation of  $B_3$ -type in the terminology of Goursat. In our case there are only two generalized flow functions, one along each characteristic. We also note that Mukminov [8] considered transformations of Monge–Ampère equations to characteristic form using generalized flow functions which depend only on first derivatives. The generalized flow functions in (7) differ from Mukminov’s through their dependence on second derivatives.

### 5. Conclusion

We have shown that there is a Bäcklund-type transformation between the Monge–Ampère equation and the sine–Gordon equation which relates the cotangent of the angle between asymptotic directions to the mean curvature of the surface. This result is sufficient to translate the infinite hierarchy of conserved quantities, auto-Bäcklund transformation and other well known properties of the sine–Gordon equation (2) to its Monge–Ampère equivalent (1).

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